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A note on "Double bounded Kumaraswamy-power series class of distributions"

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Abstract

In a recent edition of SORT, Bidram and Nekoukhou proposed a novel class of distributions and derived its mathematical properties. Several of the mathematical properties are expressed as single infinite sums or double infinite sums. Here, we show that many of these properties can be expressed in terms of known special functions, functions for which in-built routines are widely available.

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1. Introduction

Bidram and Nekoukhou (2013), referred to as BN from now, present a novel class of distributions referred to as *double bounded Kumaraswamy-power series distributions*. They derive various mathematical properties of the distributions, including their density functions, survival functions, hazard rate functions, quantiles, moment generating functions, moments, order statistic properties and stress strength parameter. They also discuss maximum likelihood estimation of the parameters of the distributions and provide a real data application.

Several of the expressions given in BN involve single infinite sums or double infinite sums. This is the case with the moment generating functions given in BN, Table 2; the moments given in BN, Table 2; the density of the ith order statistic given in BN, page 221; the rth moment of the ith order statistic given in BN, page 221; the stress-strength

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parameter given in BN, page 222; and others. This is not very convenient for practical implementation of the mathematical properties. The aim of this note is to show that many of the infinite sums and so the mathematical properties given in BN can be reduced to known special functions, functions for which in-built routines are widely available.

Let $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}_0^- = \{0,-1,-2,\ldots\}$, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers and $\mathbb{C}$ the set of complex numbers.

The closed form expressions in Section 2 involve several special functions. First is the gamma function defined by $\Gamma(a) = \int_0^\infty t^{a-1} e^{-t} \mathrm{d}t$ for $a \in \mathbb{R}^+$. The second is the polylogarithm function defined by $\mathrm{Li}_\nu(z) = \sum_{n \geq 1} n^{-\nu} z^n$ for $|z| < 1$. The third is the generalized hypergeometric function ${}_pF_q[\cdot]$ defined by

$$ {}_pF_q[z] = {}_pF_q \left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array} ; z \right] = \sum_{k=0}^\infty \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{1} $$

where $(\lambda)_\mu$ denotes the Pochhammer symbol defined by

$$ (\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \big(\mu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}\big), \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \big(\mu = n \in \mathbb{N};\ \lambda \in \mathbb{C}\big) \end{cases} \tag{2} $$

with the convention that $(0)_0 := 1$. The *Gauss hypergeometric function* ${}_2F_1(a,b:c;z)$ is the particular case of (1) for $p = 2$, $q = 1$. In the case $a,b \in \mathbb{Z}_0^-$ are negative integers, ${}_2F_1(a,b:c;z)$ becomes a polynomial $P_N(z)$ of degree $\deg(P_N) = N = \min(-a,-b)$.

The fourth is the Fox Wright generalized hypergeometric function ${}_p\Psi_q^*[\cdot]$ with p numerator parameters $a_1,\ldots,a_p$ and q denominator parameters $b_1,\ldots,b_q$, defined by (Kilbas et al., 2006, page 56)

$$ {}_p\Psi_q^* \left[\begin{array}{c} (a_1,\rho_1),\ldots,(a_p,\rho_p) \\ (b_1,\sigma_1),\ldots,(b_q,\sigma_q) \end{array} ; z \right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (a_j)_{\rho_j n}}{\prod_{j=1}^q (b_j)_{\sigma_j n}} \frac{z^n}{n!} \tag{3} $$

for $a_j \in \mathbb{C}$, $j = 1,2,\ldots,p$, $b_k \in \mathbb{C}$, $k = 1,2,\ldots,q$, $\rho_j \in \mathbb{R}^+$, $j = 1,2,\ldots,p$ and $\sigma_k \in \mathbb{R}^+$, $k = 1,2,\ldots,q$. The series in (3) converges in the whole complex z-plane when

$$ \Delta := 1 + \sum_{j=1}^q \sigma_j - \sum_{j=1}^p \rho_j > 0. $$

If $\Delta = 0$, then the series in (3) converges for $|z| < \nabla$, where

$$\nabla := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j} \right) \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j} \right).$$

The particular case of (3) for $\rho_1 = \cdots = \rho_p = 1$ and $\sigma_1 = \cdots = \sigma_q = 1$ is the generalized hypergeometric function in (1).

In-built routines for computing these special functions are widely available in packages like Maple, Matlab and Mathematica. `Gamma[z]` in Mathematica computes the gamma function, `PolyLog[ν,z]` in Mathematica computes the polylogarithm function, `HypergeometricPFQ[{a1,...,ap}, {b1,...,bq},z]` in Mathematica computes the generalized hypergeometric function, and so on. The routines allow for arbitrary precision, so computational accuracy is not an issue.

2. Closed form expressions

The closed form expressions are given by Propositions 2.1 to 2.3. Proposition 2.1 expresses $F_{i:n}(x)$, the cumulative distribution function of the ith order statistic given in BN, page 217, equation (12), in terms of the Gauss hypergeometric function. Proposition 2.2 expresses the moments given in BN, page 220, Table 2 in terms of the Fox Wright generalized hypergeometric function. These moments are to any real order, the ones given in BN were for integer orders only. Proposition 2.3 expresses R, the stress-strength parameter given in BN, page 222, in terms of the polylogarithm function.

Proposition 2.1 *For all $1 \le i \le n$ and for all $x \in (0,1)$,*

$$F_{i:n}(x) = \binom{n}{i} A^i \, {}_2F_1\left(-n+i, i; i+1; A\right) = P_n(A)$$

is a polynomial in A, where

$$A = 1 - \frac{C\left(\theta \left(1 - x^a\right)^b\right)}{C(\theta)}.$$

Moreover, $F_{n:n}(x) = A^n$.

Proof: Follows by noting

$$F_{i:n}(x) = \frac{1}{\mathrm{B}(i, n-i+1)} \sum_{k=0}^{n-i} \binom{n-i}{k} \frac{(-1)^k}{k+i} A^{k+i}$$

$$= \frac{A^i}{\mathrm{B}(i,n-i+1)} \sum_{k=0}^{n-i} \frac{(-n+i)_k \Gamma(k+i)}{\Gamma(k+i+1)} \frac{A^k}{k!}$$

$$= \frac{A^i}{i\,\mathrm{B}(i,n-i+1)} \sum_{k=0}^{n-i} \frac{(-n+i)_k (i)_k}{(i+1)_k} \frac{A^k}{k!}$$

$$= \frac{A^i\,\Gamma(n+1)}{\Gamma(i+1)\,\Gamma(i,n-i+1)}\ {}_2F_1(-n+i,i;i+1;A)$$

$$= \frac{A^i\,n!}{i!\,(n-i)!}\ {}_2F_1(-n+i,i;i+1;A).$$

The hypergeometric function reduces to 1 when $i = n$, so $F_{n:n}(x) = A^n$. ∎

Proposition 2.2 *Let X_{KG}, X_{KP}, X_{KL} and X_{KB} be random variables following, respectively, the Kumaraswamy geometric, Kumaraswamy Poisson, Kumaraswamy logarithmic and Kumaraswamy binomial distributions defined in BN. Then, for all real $r > -a$ and $b > 0$, we have*

$$\mathbb{E}\big(X_{\mathrm{KG}}^r\big) = b(1-\theta)\mathrm{B}\left(1+\frac{r}{a},b\right)\ {}_2\Psi_1^*\left[\begin{matrix}(1+b,b),\ (1,1)\\ \left(1+b+\frac{r}{a},b\right)\end{matrix};\theta\right], \tag{4}$$

$$\mathbb{E}\big(X_{\mathrm{KP}}^r\big) = \frac{b\,\theta}{e^\theta-1}\,\mathrm{B}\left(1+\frac{r}{a},b\right)\ {}_1\Psi_1^*\left[\begin{matrix}(b,b)\\ \left(1+b+\frac{r}{a},b\right)\end{matrix};\theta\right], \tag{5}$$

$$\mathbb{E}\big(X_{\mathrm{KL}}^r\big) = -\frac{b\theta}{\log(1-\theta)}\mathrm{B}\left(1+\frac{r}{a},b\right)\ {}_2\Psi_1^*\left[\begin{matrix}(b,b),\ (1,1)\\ \left(1+b+\frac{r}{a},b\right)\end{matrix};\theta\right] \tag{6}$$

and

$$\mathbb{E}\big(X_{\mathrm{KB}}^r\big) = -\frac{bm\theta}{(1+\theta)^m-1}\mathrm{B}\left(1+\frac{r}{a},b\right)\ {}_2\Psi_1^*\left[\begin{matrix}(b,b),\ (1-m,1)\\ \left(1+b+\frac{r}{a},b\right)\end{matrix};-\theta\right]. \tag{7}$$

Each of these expressions is valid for all $|\theta| < 1$.

Proof: (4) follows by noting that

$$\begin{aligned}
\mathbb{E}\big(X_{\mathrm{KG}}^r\big) &= b(1-\theta)\sum_{n\geq 1} n\,\mathrm{B}\left(1+\frac{r}{a}, nb\right)\,\theta^{n-1} \\
&= b(1-\theta)\Gamma\left(1+\frac{r}{a}\right)\sum_{n\geq 1}\frac{n\,\Gamma(nb)\,\theta^{n-1}}{\Gamma\left(1+\frac{r}{a}+nb\right)} \\
&= (1-\theta)\Gamma\left(1+\frac{r}{a}\right)\sum_{n\geq 1}\frac{\Gamma(nb+1)\,\theta^{n-1}}{\Gamma\left(1+\frac{r}{a}+nb\right)} \\
&= (1-\theta)\Gamma\left(1+\frac{r}{a}\right)\sum_{m\geq 0}\frac{\Gamma(1+b+mb)\,\theta^{m}}{\Gamma\left(1+b+\frac{r}{a}+mb\right)} \\
&= (1-\theta)\frac{\Gamma\left(1+\frac{r}{a}\right)\Gamma(1+b)}{\Gamma\left(1+b+\frac{r}{a}\right)}\sum_{m\geq 0}\frac{(1+b)_{mb}\,(1)_m}{\left(1+b+\frac{r}{a}\right)_{mb}}\frac{\theta^m}{m!}
\end{aligned}$$

and that the infinite sum in the last step corresponds to a Fox Wright generalized hypergeometric function with $\Delta = 0$, $\nabla = 1$.

(5) follows by noting that

$$\begin{aligned}
\mathbb{E}\big(X_{\mathrm{KP}}^r\big) &= \frac{b}{e^\theta - 1}\sum_{n\geq 1}\mathrm{B}\left(1+\frac{r}{a}, nb\right)\,\frac{\theta^n}{(n-1)!} \\
&= \frac{b\,\Gamma\left(1+\frac{r}{a}\right)\,\theta}{e^\theta - 1}\sum_{m\geq 0}\frac{\Gamma(b+mb)}{\Gamma\left(1+b+\frac{r}{a}+mb\right)}\frac{\theta^m}{m!} \\
&= \frac{b\,\Gamma(b)\,\Gamma\left(1+\frac{r}{a}\right)\,\theta}{\left(e^\theta - 1\right)\,\Gamma\left(1+\frac{r}{a}+b\right)}\sum_{m\geq 0}\frac{(b)_{mb}}{\left(1+b+\frac{r}{a}\right)_{mb}}\frac{\theta^m}{m!}
\end{aligned}$$

and that the infinite sum in the last step corresponds to a Fox Wright generalized hypergeometric function with $\Delta = 1$.

The proof of (6) is similar to the proof of (4).

(7) follows by noting that

$$\begin{aligned}\mathbb{E}\left(X_{\mathrm{KB}}^{r}\right) &= \frac{b\theta}{(1+\theta)^{m}-1}\sum_{n\geq 1} n\binom{m}{n}\,\mathrm{B}\left(1+\frac{r}{a},nb\right)\,\theta^{n-1}\\ &= \frac{b\theta m}{(1+\theta)^{m}-1}\,\Gamma\left(1+\frac{r}{a}\right)\sum_{n-1\geq 0}\frac{(-1)^{n-1}(1-m)_{n-1}\Gamma(b+(n-1)b)}{\Gamma\left(1+b+\frac{r}{a}+(n-1)b\right)}\frac{\theta^{n-1}}{(n-1)!}\\ &= \frac{b\theta m}{(1+\theta)^{m}-1}\,\frac{\Gamma(b)\Gamma\left(1+\frac{r}{a}\right)}{\Gamma\left(1+b+\frac{r}{a}\right)}\sum_{k\geq 0}\frac{(-1)^{k}(1-m)_{k}(b)_{kb}}{\left(1+b+\frac{r}{a}\right)_{kb}}\frac{\theta^{k}}{k!}\end{aligned}$$

and that the infinite sum in the last step corresponds to a Fox Wright generalized hypergeometric function with $\Delta = 0$, $\nabla = 1$. ■

Proposition 2.3 *For all* $|\theta_1| < 1$ *and* $|\theta_2| < 1$,

$$\begin{aligned}R(\theta_1,\theta_2) &= \sum_{k\geq 0}\sum_{j\geq 0}\frac{(k+1)\theta_1^{k}\theta_2^{j}}{(k+j+1)^{2}(k+j+2)}\\ &= \frac{1}{(\theta_2-\theta_1)^{2}}\Big[\mathrm{Li}_2(\theta_1)-\mathrm{Li}_2(\theta_2)+\theta_1-\theta_2\\ &\qquad +(2-\theta_2/\theta_1-\theta_2)\,\log(1-\theta_1)-(1-\theta_2)\,\log(1-\theta_2)\Big]. \qquad (8)\end{aligned}$$

Proof: The double series for $R(\theta_1,\theta_2)$ converges for all $|\theta_1| < 1$, $|\theta_2| < 1$, where it is term-wise differentiable with respect both θ_1, θ_2. To obtain a closed form expression for R, we define

$$\mathfrak{R}(\theta_1,\theta_2) = \sum_{k\geq 0}\sum_{j\geq 0}\frac{\theta_1^{k}\theta_2^{j}}{(k+j+1)^{2}(k+j+2)}.$$

By the differentiability property,

$$R = \frac{\mathrm{d}}{\mathrm{d}\theta_1}\left(\theta_1\cdot\mathfrak{R}(\theta_1,\theta_2)\right). \qquad (9)$$

By partial fractions,

$$\mathfrak{R}(\theta_1,\theta_2)=\sum_{k\geq 0}\sum_{j\geq 0}\left(\frac{1}{k+j+2}-\frac{1}{k+j+1}+\frac{1}{(k+j+1)^2}\right)\,\theta_1^k\,\theta_2^j=:S_1-S_2+S_3.\ (10)$$

By legitimate integration-summation order exchange and the definition of gamma function,

$$S_1=\int_0^\infty e^{-2t}\sum_{k\geq 0}\left(\theta_1e^{-t}\right)^k\sum_{j\geq 0}\left(\theta_2e^{-t}\right)^j\,\mathrm{d}t=\int_0^\infty\frac{e^{-2t}\,\mathrm{d}t}{\left(1-\theta_1e^{-t}\right)\left(1-\theta_2e^{-t}\right)},$$

$$S_2=\int_0^\infty e^{-t}\sum_{k\geq 0}\left(\theta_1e^{-t}\right)^k\sum_{j\geq 0}\left(\theta_2e^{-t}\right)^j\,\mathrm{d}t=\int_0^\infty\frac{e^{-t}\,\mathrm{d}t}{\left(1-\theta_1e^{-t}\right)\left(1-\theta_2e^{-t}\right)},$$

$$S_3=\int_0^\infty te^{-t}\sum_{k\geq 0}\left(\theta_1e^{-t}\right)^k\sum_{j\geq 0}\left(\theta_2e^{-t}\right)^j\,\mathrm{d}t=\int_0^\infty\frac{t\,e^{-t}\,\mathrm{d}t}{\left(1-\theta_1e^{-t}\right)\left(1-\theta_2e^{-t}\right)}.$$

Routine but lengthy calculations show that

$$S_1=\frac{\theta_1\log(1-\theta_2)-\theta_2\log(1-\theta_1)}{\theta_1\theta_2(\theta_1-\theta_2)},$$

$$S_2=\frac{\log(1-\theta_2)-\log(1-\theta_1)}{\theta_1-\theta_2}.$$

Using the fact

$$\int_0^\infty\frac{t\,e^{-t}\,\mathrm{d}t}{1-ae^{-t}}=\frac{1}{a}\,\mathrm{Li}_2(a),$$

we can reduce

$$S_3=\frac{1}{\theta_2-\theta_1}\int_0^\infty\frac{t\,e^{-t}\,\mathrm{d}t}{1-\theta_1e^{-t}}-\frac{1}{\theta_2-\theta_1}\int_0^\infty\frac{t\,e^{-t}\,\mathrm{d}t}{1-\theta_2e^{-t}}=$$

$$=\frac{1}{\theta_2-\theta_1}\left[\frac{1}{\theta_1}\,\mathrm{Li}_2(\theta_1)-\frac{1}{\theta_2}\,\mathrm{Li}_2(\theta_2)\right].$$

Collecting $S_j, j = 1,2,3$, we obtain by virtue of (10) that

$$\mathfrak{R}(\theta_1,\theta_2) = \frac{\theta_2(1-\theta_1)\log(1-\theta_1) - \theta_1(1-\theta_2)\log(1-\theta_2)}{\theta_1\theta_2(\theta_2-\theta_1)}$$

$$+ \frac{1}{\theta_2-\theta_1}\left[\frac{1}{\theta_1}\,\mathrm{Li}_2(\theta_1) - \frac{1}{\theta_2}\,\mathrm{Li}_2(\theta_2)\right]$$

$$= \frac{(1-\theta_1)\log(1-\theta_1) + \mathrm{Li}_2(\theta_1)}{\theta_1(\theta_2-\theta_1)} - \frac{(1-\theta_2)\log(1-\theta_2) + \mathrm{Li}_2(\theta_2)}{\theta_2(\theta_2-\theta_1)}.$$

The result follows by applying (9). ■

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