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Robust project management with the tilted beta distribution

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Abstract

Recent years have seen an increase in the development of robust approaches for stochastic project management methodologies such as PERT (Program Evaluation and Review Technique). These robust approaches allow for elevated likelihoods of outlying events, thereby widening interval estimates of project completion times. However, little attention has been paid to the fact that outlying events and/or expert judgments may be asymmetric. We propose the tilted beta distribution which permits both elevated likelihoods of outlying events as well as an asymmetric representation of these events. We examine the use of the tilted beta distribution in PERT with respect to other project management distributions.

MSC: 62E15, 90B99, 62P30

Keywords: Activity times, finite mixture, PERT, tilted beta distribution, robust project management, sensitivity analysis.

1. Introduction

In project management it is important to be able to assess the total time for a project's completion. Since projects can be very complex, methodologies such as the Program Evaluation and Review Technique (PERT) (Malcolm et al., 1959) have been developed to assist in these assessments. PERT has been used for many decades but in recent years academics, managers, and policy makers have increasingly realized that conventional modeling approaches and tools may not be well-equipped to deal with extreme events. For example, few lenders would have predicted that the rise of lending to the sub-prime

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market in the United States would cost them their own jobs, and fewer still would have predicted that it would have global repercussions. Hence there is a growing appreciation of the need for more robust models that assign greater probability to more extreme events.

Recent research by authors such as Hahn (2008) and López Martín et al. (2012) has described some ways for increasing the amount of distributional uncertainty in the context of project management tools such as PERT. The goal of this research stream has been to extend the PERT framework to accommodate greater likelihood to extreme tail-area outcomes. This has led to the ability to provide wider confidence intervals for activity and project duration times and hence more conservative results, while still retaining the classic PERT results as an important special case. The ability to increase distributional uncertainty is an important first step towards robust project management estimation; however, one consideration that has been underexplored is that one extreme may be more likely or more important than another. For example, as documented below, project managers tend to provide positively biased time estimates. Accordingly, project management tools which depend on these biased estimates are likely to underpredict the overall project time. In the current paper, we describe a new distribution that can be used by an independent agent (such as a risk manager) to differentially weight high versus low extremes. This can be used to help counteract some biased estimates.

This paper is structured as follows. Firstly, we review the literature about alternative distributions used in the area of PERT methodology. In Section 3 we present the tilting distribution, as a particular case of generalized Topp and Leone distribution; the tilted beta distribution, as a mixture between the tilting and beta distributions, and some stochastic characteristics. The elicitation for the distribution is presented in Section 4. The results are illustrated with an example in Section 5. Finally, Section 6 summarizes the main conclusions.

2. Literature

Projects often fail to meet various financial and scheduling targets despite management's best efforts to ensure success. For example, Bevilacqua et al. (2009) report on budget overruns and non-completion of tasks in projects undertaken with the use of PERT methodologies in the energy sector. Hence, there have been numerous studies which have tried to understand the sources of the persistent problem of project management overestimates or underestimates. Boulding et al. (1997) find that senior level executive subjects tended to ignore negative information or distort the information to fit preconceived notions and decisions. Hill et al. (2000) find that expert project managers sometimes overestimate and sometimes underestimate project durations, but that the underestimates were greater in magnitude leading on average to underestimation. Keil et al. (2007) conducted a laboratory experiment which revealed that failure to recognize problems early also leads to over-optimistic assessments regarding information technology projects. Snow et al. (2007) describe an in-depth research program on the assessment of biases among software development project managers. The most commonly reported reason for giving optimistic judgments was to avoid being the bearer of bad news. Of the 56 surveyed project managers, 22% mentioned providing optimistic judgments because of a belief that senior management "shoots the messenger" while another 22% indicated that optimistic judgments were provided so as to make the project manager look good. Project managers were also two times more likely to be optimistically biased than they were to be pessimistically biased. Snow et al. (2007) conclude "optimistic bias leads to status reports that are very different from reality, while pessimistically biased status reports tend to be accurate because bias offsets error". Iacovou et al. (2009) also found that optimistically-biased reports were more prevalent than pessimistically biased ones in a sample of 390 information systems project managers, a finding consistent with work by Smith et al. (2001) and Gillard (2005). In a related vein, project managers who are able to accurately assess the risks of a troubled project are more likely to discontinue the project (Keil et al., 2000). Sengupta et al. (2008) conducted research on several hundred project managers and found that managers seem to strongly anchor on the initial risk assessment, and find it difficult to update their opinion with new information that should have prompted a re-assessment. One of the mitigation strategies identified by Sengupta et al. (2008) was better calibration of forecasting tools to project particulars.

Given the large volume of research which indicates project managers may tend toward having an optimistic bias, one possible solution is to provide a system whereby a third party (such as a risk manager) can provide an outside independent review to help remove bias in estimates. Öztaş and Ökmen (2005) describe a project management methodology called the judgmental risk analysis process that is explicitly pessimistic in nature. This is implemented by assessing a separate risk factor for each activity and assigning a probability distribution to the risk factor. The minimum and maximum activity times are then modified by including additive and subtractive offsets based on the activity risk factor to these activity times. Here we observe that if managers or other experts tend to overemphasize optimistic information, then counteracting this is a matter of de-emphasizing or downweighting optimistic information. In the current paper, we provide a probabilistic approach that permits a risk manager to introduce a negative (or positive) weighting across the range that nonetheless retains the usual PERT framework as a special case. This is accomplished by the introduction of a new distribution called the tilted beta distribution. In the following section, we present this distribution and explore some of its main properties which are relevant for project management. We then present an application of this distribution and conclude with discussion.

3. Distributions for project activity times

Malcolm et al. (1959) were the first to use the beta distribution to describe project management activity times. The beta distribution is the most prevalent distribution used in stochastic project management due to its useful properties and appearance in the seminal work of Malcolm et al. (1959). Other widely used continuous probability models within the PERT methodology are the triangular distribution (Clark, 1962; MacCrimmon and Ryavec, 1964; Moder and Rodgeres, 1968; Vãduva, 1971; Megill, 1984; Williams, 1992; Keefer and Verdini, 1993; Johnson, 1997), the trapezoidal distribution (Pouliquen, 1970; Herrerías and Calvete, 1987; Herrerías, 1989; Powell and Wilson, 1997; Garvey, 2000), the doubly truncated normal distribution (Kotiah and Wallace, 1973), the uniform distribution (Suárez, 1980; Romero, 1991), the generalized biparabolic distribution (García et al., 2010) and the Parkinson distribution (Trietsch et al., 2012).

More recently, the literature on distributions for project management activity times has emphasized the importance of accounting for heavy tails and assigning more probability density to extreme values (Mohan et al., 2007). In addition, the research emphasis has moved away from using 'off the shelf' statistical distributions and instead has sought to engineer new distributions that are tailored to satisfy PERT desiderata. For example Hahn (2008) proposed the beta rectangular distribution which is a bounded distribution like the beta but assigns greater density to extreme values and can accommodate very heavy tails. Similarly, García et al. (2010) presented the generalized biparabolic distribution and demonstrated its capacity to have larger variances than the beta distribution. Our motivation in writing this paper is to engineer a distribution to address the documented optimistic bias discussed above while also addressing the need for heavy tails and large variances which has been identified previously.

3.1. Beta distribution

We begin with a brief review of the beta distribution given its importance in both the current work and in the project management literature. The standard beta distribution defined on $[0,1]$ has the following probability density function (pdf)

$$p(x|\alpha,\beta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

It is necessary that both $\alpha > 0$ and $\beta > 0$ for (1) to be a valid pdf. The mean of (1) is $\alpha/(\alpha+\beta)$ while the variance is $(\alpha\beta)/\left((\alpha+\beta)^2(\alpha+\beta+1)\right)$.

The beta distribution is capable of a variety of shapes (see distributions having dotted lines in Figure 3). Unfortunately the beta distribution does not provide a great deal of flexibility when it is of interest to preserve the typically-preferred unimodal shapes but assign higher probability to extremal (or 'tail area') events. This observation led Hahn (2008) to propose the beta rectangular distribution, which is a mixture of the beta and rectangular distributions, for project management activity times and it is defined by

$$p(x|\alpha,\beta,\theta) = \begin{cases} \theta \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} + (1-\theta) & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

where θ is a mixing parameter such that $0 \le \theta \le 1$.

Under the PERT conditions (to be discussed in Section 4), the beta rectangular distribution permits larger variances than the beta distribution and allows for elevated tail-area density (see Figure 1. The beta rectangular also has the beta distribution as a special case; hence, the classic PERT activity time parameters can be easily obtained as a particular case.

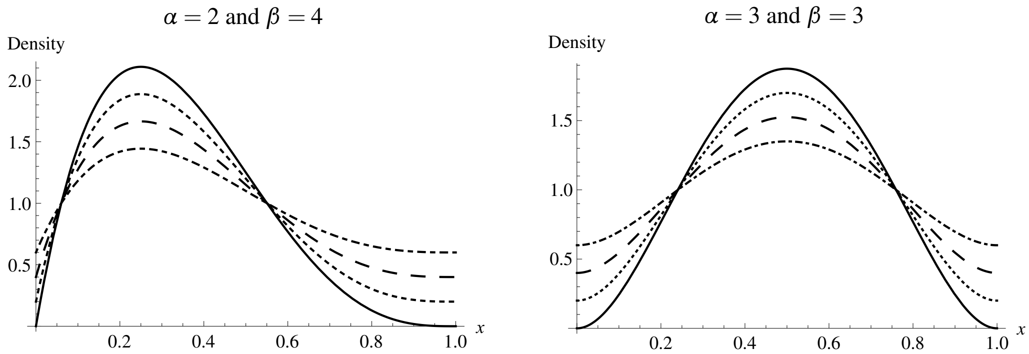


Figure 1: *Examples of beta rectangular distribution for $\theta = 1$ (solid), $\theta = 0.8$ (dotted), $\theta = 0.6$ (dashed) and $\theta = 0.4$ (dash-dotted).*

However, the previous discussion of Section 2 indicates that project managers may have an optimistic bias and the beta rectangular does not provide a way for addressing this issue. The remainder of this section is dedicated to formulating a distribution that addresses this issue. Accordingly next we describe the tilting distribution which allows for a straightforward way of expressing an optimistic (or pessimistic) bias. This will in turn allow us to construct the tilted beta distribution whereupon we will study in depth the characteristics of this distribution.

3.2. Tilting distribution

Topp and Leone (1955) present a distribution with probability density function (pdf) defined by $f(x,\beta) = \beta(2-2x)(2x-x^2)^{\beta-1}$, where $x \in [0,1]$ and $\beta > 0$. Depending on the values of β the distribution either has a J-shaped form ($0 < \beta < 1$); is unimodal ($\beta > 1$); or is left-triangular ($\beta = 1$). Kotz and van Dorp (2004) present a generalization of the Topp and Leone distribution by considering a slightly more general generating pdf, whose expression is

$$g(x|\alpha) = \alpha - 2(\alpha - 1)x, \tag{3}$$

where α defined on the interval $[0,2]$. The authors define the slope distributions as the distributions with pdf of the form (3).

Taking the reparametrization $\alpha = 2v$, we introduce a distribution, called the tilting distribution, which has the following density function:

$$p(x|v) = \begin{cases} 2v - 2(2v-1)x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

As $0 \leq \alpha \leq 2$, the parameter v is defined on the interval $[0,1]$. The reparameterization considered here leads to a parameter range consistency for the tilted beta, as shown later.

The cumulative density function (CDF) of (4) is

$$F(x|v) = \begin{cases} 0 & \text{if } x < 0, \\ 2vx - (2v-1)x^2 & \text{if } 0 \leq x \leq 1, \\ 1 & \text{if } x > 1. \end{cases} \tag{5}$$

Graphical examples of density and cumulative density function of the tilting distribution are shown in Figure 2. Figure 2 reveals that when $v = 1/2$ the uniform distribution is obtained and when $v = 0$ or $v = 1$, a triangular distribution with mode v is obtained.

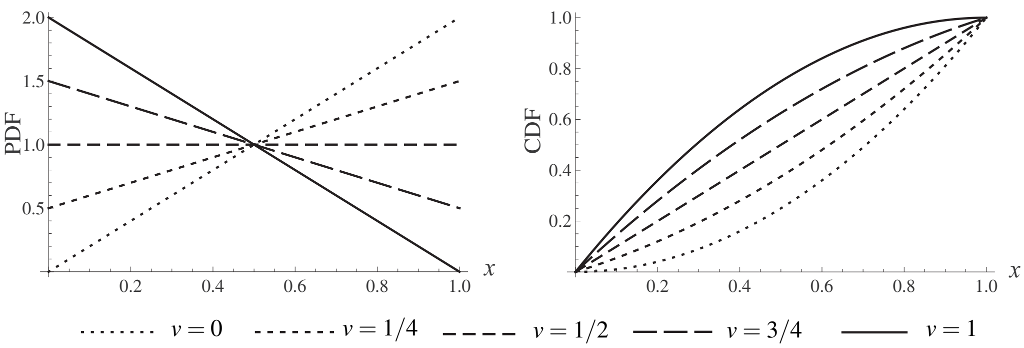


Figure 2: *Examples of PDF and CDF of tilting distributions.*

The mean, variance and coefficient of skewness of the tilting distribution are respectively

$$E(X) = \frac{2-v}{3}, \qquad \text{var}(X) = \frac{2v(1-v)+1}{18}, \qquad \beta_1 = \frac{2\sqrt{2}\left(1+3v-15v^2+10v^3\right)}{5\left(1+2v-2v^2\right)^{3/2}}. \tag{6}$$

Taking the first derivative of $\mathrm{var}(X)$ with respect to v the variance of the distribution is maximized for $v = 1/2$. When $v = 0$ or $v = 1$ the variance of the distribution is minimum whilst the coefficient of skewness is maximum.

In some contexts it can be necessary to work with a variable defined over the more general support $[a,b]$ instead of $[0,1]$. For cases where the variable may take on an arbitrary location and scale, we describe the variable $Y = a + (b-a)X$ with $b > a$. The inverse function is $X = \frac{y-a}{b-a}$ with Jacobian $\partial X/\partial y = \frac{1}{b-a}$. Then, the density function of (4) with to support $[a,b]$ is

$$p(y|w,a,b) = \frac{2}{b-a}\begin{cases} \frac{w-a}{b-a} - \left(2\frac{w-a}{b-a} - 1\right)\left(\frac{y-a}{b-a}\right) & \text{if } a \le y \le b, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

where $w = a + (b-a)v$. The quantile function of Y is

$$P^{-1}(q|w) = \begin{cases} \frac{a(2b-w)-bw+(b-a)\sqrt{a^2(1-q)+b^2q-2aw(1-q)-2bwq+w^2}}{a-2w+b} & \text{if } w \neq \frac{a+b}{2}, \\ a+(b-a)q & \text{if } w = \frac{a+b}{2}, \end{cases} \tag{8}$$

with $0 < q < 1$.

Although the introduction of additional parameters is associated with an increased complexity for the distributional expressions, the increase in flexibility makes it worthwhile to briefly summarize a few key expressions. In this case, the mean and variance of the tilting distribution are

$$E[Y] = \frac{2a-w+2b}{3}, \quad \mathrm{var}[Y] = \frac{a^2+2(a+b)w+b^2}{6} - \frac{(a+w+b)^2}{9}. \tag{9}$$

3.3. Tilted beta distribution

Having presented a few key properties of (4), we now introduce the tilted beta distribution. The density function of a random variable X having the tilted beta distribution with $\alpha > 0$, $\beta > 0$, $v \in [0,1]$, and $\theta \in [0,1]$ is

$$p(x|v,\alpha,\beta,\theta) = \begin{cases} (1-\theta)\left[2v-2(2v-1)x\right] + \theta\left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}\right] & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

This distribution can be seen as a mixture of the tilting and beta distribution. The parameter θ indicates the relative proportionality of the tilting distribution to the beta distribution and v can be interpreted as the relative tilt proportionality. When $\theta = 1$

the beta distribution is obtained, for $\theta = 0$ we obtain the tilting distribution of (4) and $\theta = 1/2$ indicates a balance between the two distributions. With respect to the parameter v, $v = 0$ indicates the maximum downward tilt, $v = 1$ indicates maximum upward tilt, and $v = 1/2$ indicates a balance of upward and downward tilt. Figure 3 shows that the beta distribution, the uniform distribution, and the beta rectangular (Hahn, 2008) are all special cases of the distribution (10). Indeed, the density of the resulting tilted beta keeps the property of smoothness possessed by the beta distribution. This property can be contrasted with discontinuous or 'sharp' distributions that have been proposed for PERT such as the triangular (Johnson, 1997) and its extensions in the two-sided power distribution family (García Pérez et al., 2005; Herrerías et al., 2009).

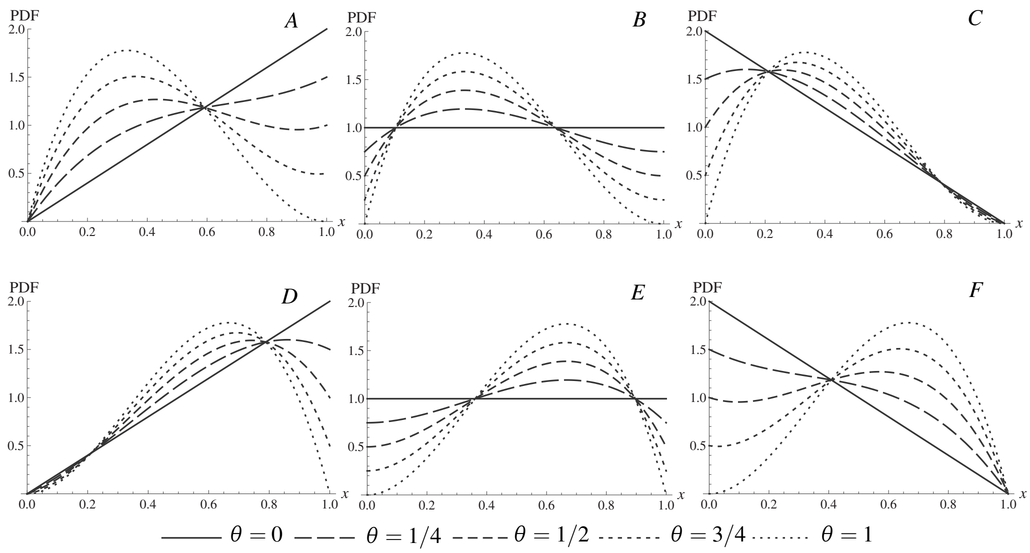


Figure 3: *Examples of tilted beta distributions for:* $\alpha = 2, \beta = 3, v = 0$ *(A);* $\alpha = 2, \beta = 3, v = 0.5$ *(B),* $\alpha = 2, \beta = 3, v = 1$ *(C),* $\alpha = 3, \beta = 2, v = 0$ *(D);* $\alpha = 3, \beta = 2, v = 0.5$ *(E),* $\alpha = 3, \beta = 2, v = 1$ *(F).*

The moment generating function of (10) is defined by

$$M_x(t) = 2\frac{e^t\,(t-1+(2-t)v)+1-(2+t)v}{t^2}(1-\theta) + {}_1F_1[\alpha, \alpha+\beta, t]\theta \qquad (11)$$

where ${}_1F_1$ indicates the Kummer confluent hypergeometric function. From (11) one can obtain the mean and the second moment of the tilted beta distribution

$$E(X) = (1-\theta)\frac{2-v}{3} + \theta\frac{\alpha}{\alpha+\beta}, \qquad (12)$$

$$E(X^2) = (1-\theta)\frac{3-2v}{6} + \theta\frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)}. \qquad (13)$$

We can consider the density of the pdf (10) at the endpoints making the usual assumption that α and β are both greater than or equal to 1. When $x=0$, it can be shown that $p(x|v,\alpha,\beta,\theta)=(1-\theta)2v$. When $x=1$, $p(x|v,\alpha,\beta,\theta)=(1-\theta)(2-2v)$. Observe that the density at the endpoints will be different as long as $v\neq 0.5$ and $\theta\neq 1$. As will be mentioned later, very few distributions applied in the PERT methodology have this property.

In the existing literature on this issue, few distributions exhibit a shape similar to that of the tilted beta. One such distribution is the elevated two-sided power distribution introducing by García et al. (2011). However, here we respectfully note that the elevated two-sided power distribution requires an additional parameter and possesses more complex expressions (cf. the simplicity of (13) above versus equation (23) of García et al. (2011)). In contrast, the current distribution is a mixture of two tractable distributions and hence it is straightforward to implement in any environment where the beta distribution of PERT has been previously applied. The reflected generalized Topp and Leone distribution (Van Dorp and Kotz, 2006) can achieve somewhat similar shapes but it does not seem possible for this distribution to have appreciable density at both extremes simultaneously. Thus, this distribution is not well-suited for circumstances where tail-area events have appreciable likelihood at both high and low extremes. Moreover, Van Dorp and Kotz (2006) indicate that this distribution does not have closed-form moment expressions except for special cases (cf. with (13) above), again adding computational cost for applications-oriented Monte Carlo simulation. Finally, Pham-Gia and Turkkan (1993) presented an explicit expression for the distribution of the difference of two beta distributions. This distribution can also take on many flexible shapes (Nadarajah and Gupta, 2004, see pp. 71–84). However, it also has a complex specification and, for example, its moments can only be analytically approximated.

Note that the procedure to raise any bounded continuous distribution by the tilting distribution is equivalent to the procedure of raising the density of the distribution linearly, and then re-normalizing.

4. Elicitation of the tilted beta distribution's parameters

In many project management applications, it is necessary to consider parameter elicitation for distributions. Direct elicitation of the beta distribution's α and β is always an option (e.g., Chaloner and Duncan, 1983). Historically however project management applications have used the classic PERT parameters: a (lower bound), m (most likely) and b (upper bound). The classic PERT formulas are then

$$E(Y)=\frac{a+4m+b}{6}, \tag{14}$$

$$\mathrm{V}(Y)=\frac{(b-a)^2}{36}. \tag{15}$$

A wide literature has been dedicated to examining the necessary conditions linking (14) and (15) to the parameters of the beta distribution (Malcolm et al., 1959; Clark, 1962; Grubbs, 1962; Sasieni, 1986; Gallagher, 1987; Littlefield and Randolph, 1987; Kamburowski, 1997). To summarize, (14) holds exactly when $k = \alpha + \beta = 6$ and $\alpha \neq \beta$. We may call this the Type I beta condition. Further, (14) and (15) simultaneously hold when: $\alpha = \beta = 4$; $\alpha = 3+\sqrt{2}$, $\beta = 3-\sqrt{2}$; and $\alpha = 3-\sqrt{2}$, $\beta = 3+\sqrt{2}$ (Grubbs, 1962). We may call this the Type II beta condition. Clearly the Type II condition is more restrictive than the Type I condition. In this case, all that is required is to select whether a symmetric, positively skewed, or negatively skewed distribution is required. Then the values of α and β are given as above. For the Type I condition, note that the mean and mode of the beta distribution are α/k and $(\alpha-1)/(k-2)$, respectively. Hence, solving simultaneous equations for the mean and the mode we find in the case of a standardized beta ($a = 0$ and $b = 1$) that the values of α and β under the Type I condition are $\alpha = 4m+1$ and $\beta = 5-4m$.

Having addressed the elicitation of α and β, we turn to the elicitation of the remaining parameters of (10). The elicitation of the mixture parameter θ has been considered by Hahn (2008) and López Martín et al. (2012) using the parameter λ. Hence it remains to discuss v. Eliciting v can be accomplished by the following procedure. We assume the expert believes there is a linear increase or decrease in the probability density across time in accordance with the shape of the tilting distribution. Let A_j represent the event that a particular activity is completed on day j. Then we ask the expert to provide the probability of the event of activity completion in day j. This is denoted by $p(A_j)$. Next we ask her to give the probability of the event of activity completion in day $j+1$, which is denoted by $p(A_{j+1})$. Suppose a discrete approximation to the tilting distribution is used. The slope is (see Figure 4)

$$\frac{p(A_{j+1}) - p(A_j)}{(j+1) - j} \tag{16}$$

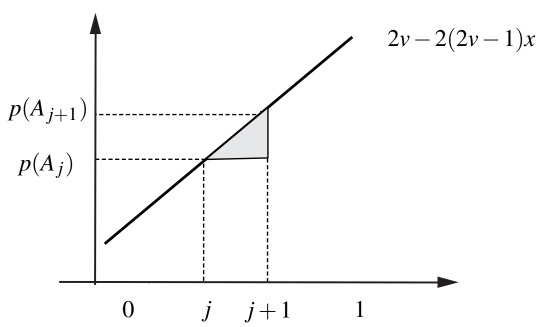


Figure 4: *Cumulative distribution function of a discrete variable with support* $[0,1]$.

by the definition of the slope as $\frac{\mathsf{y}_2-\mathsf{y}_1}{\mathsf{x}_2-\mathsf{x}_1}$. Since there are $b-a$ conceivable activity completion days, we may normalize the cumulative activity time until A_{j+1} as $\mathsf{x}_2 = \dfrac{j+1}{b-a}$. Similarly we may normalize the cumulative activity time until A_j as $\mathsf{x}_1 = \dfrac{j}{b-a}$. Since A_{j+1} and A_j differ by one day out of the $b-a$ total activity days, we substitute into the slope formula to define the rate of change as

$$r = \frac{p(A_{j+1}) - p(A_j)}{1/(b-a)}. \tag{17}$$

Note that other time units besides days may be alternatively used. Once we have obtained r, we can solve for the value of v by making r equal to the slope of the density function $-2(2v-1)$ and solving in terms of v, yielding

$$v = \frac{2-r}{4}. \tag{18}$$

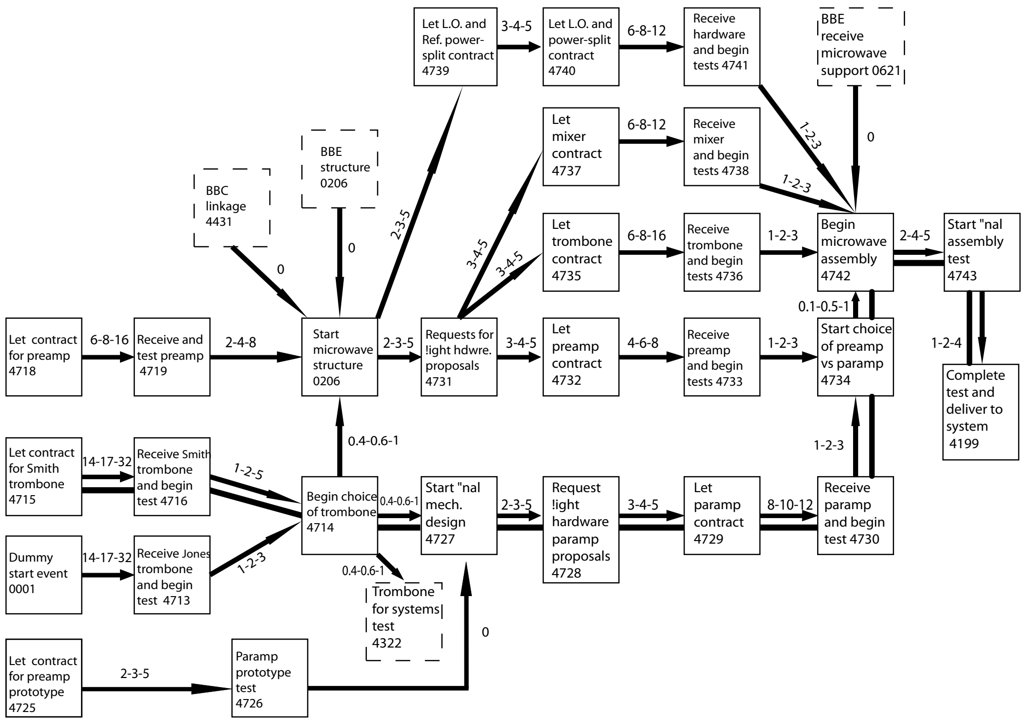
If $v \notin [0,1]$, then a re-examination will be required. Discussion with the expert can be undertaken to reveal whether, for example, the judgment task can be made easier by considering months instead of days. Alternatively, it may be that a linearly-sloped distribution does not correspond to the expert beliefs and if so the process would need to be terminated. Assuming a valid value of v, conversion to w is given by $w = a + (b-a)v$.

Alternatively, we ask the expert the probability of the event of activity completion in day $j+1, j+2, \ldots, j+k$, where the period $j+1$ is the following day of the first day after the start of the project and $j+k$ is the day before the end of the project. For each probability, and using the expression (18), we elicit the parameter v for each different day. For example we obtain $v_1, v_2, \ldots, v_k$. We can then find v as the arithmetic mean $v = \frac{1}{k}\sum_{i=1}^{k} v_i$.

Another more informal approach to elicitation of v can be contemplated by analogy with the elicitation of θ. Note in the beta rectangular distribution that $\theta = 0$ corresponds to the case of no (or 0%) additional uncertainty above and beyond that of the beta. Further, $\theta = 1$ corresponds to the case of complete (or 100%) uncertainty. So $v = 1$ would correspond to a 100% linear pessimistic belief or worst-case linear belief about the project activity completion time. In contrast, $v = 0$ would correspond to a 0% linear pessimistic belief (100% linear optimistic belief) or a best-case linear belief about the project activity completion time. More moderate values of v would represent various compromises between these extremes, with $v = 1/2$ representing neither pessimism nor optimism. In the event that we would want to counteract an elicited value v, we simply invert the slope by using $(1-v)$ in the place of v.

5. Application

Figure 5 shows 29 activities in a real-world electronic module development project from Moder et al. (1983). The critical path is marked by a heavy line.



Project activities and paths

Figure 5: *Reproduction of PERT network of an electronic module development project (Moder et al., 1983).*

The distribution for the total project time can be found using Monte Carlo simulation and accounting for the diagram's precedence relationships. To obtain our results, we simulated from the activity times using the beta distribution information from Figure 5 and the listed values of θ and v appearing in our results. The results are based on 10000 Monte Carlo simulations from the distributions of interest. Please observe that we report results arising from use of the beta rectangular distribution and, for completeness, use of the tilted beta distribution with $v = 1/2$ which is equivalent to the beta rectangular. Results for the two equivalent distributions are equivalent up to Monte Carlo error at the third significant digit with a few exceptions that are slightly larger such as the upper 95% confidence interval for $\theta = 1/4$ in Table 1 (62.34 versus 62.13).

With increasing θ, the distributions approach the beta distribution and they equal the beta when $\theta = 1$. Therefore we see in Table 1 that the standard deviation declines with

Table 1: *Stochastic characteristics of the total project time variable obtained by Monte Carlo simulations where Beta-R is the beta rectangular distribution and T-Beta is the tilted beta distribution.*

θ	Distribution	Mean	Stand. Dev.	Skewness	Kurtosis	95% C.I.
$\theta = 1/4$	Beta-R	51.50	5.55	0.23	2.26	(42.21, 62.34)
	T-Beta ($v = 0$)	56.56	5.42	-0.17	2.15	(46.41, 65.49)
	T-Beta ($v = 1/4$)	54.08	5.65	0.02	2.11	(44.13, 64.10)
	T-Beta ($v = 1/2$)	51.51	5.56	0.20	2.22	(42.12, 62.13)
	T-Beta ($v = 3/4$)	48.85	5.14	0.46	2.58	(40.64, 59.80)
$\theta = 1/2$	Beta-R	50.15	5.09	0.47	2.54	(42.05, 60.96)
	T-Beta ($v = 0$)	53.54	5.49	0.21	2.14	(44.38, 63.92)
	T-Beta ($v = 1/4$)	51.86	5.38	0.36	2.30	(43.17, 62.59)
	T-Beta ($v = 1/2$)	50.11	5.10	0.48	2.54	(42.18, 60.88)
	T-Beta ($v = 3/4$)	48.54	4.78	0.54	2.77	(41.03, 59.12)
$\theta = 3/4$	Beta-R	48.88	4.48	0.67	3.10	(41.96, 59.33)
	T-Beta ($v = 0$)	50.56	4.90	0.60	2.74	(43.09, 61.51)
	T-Beta ($v = 1/4$)	49.82	4.70	0.65	2.93	(42.63, 60.57)
	T-Beta ($v = 1/2$)	48.87	4.44	0.69	3.17	(41.95, 59.23
	T-Beta ($v = 3/4$)	48.09	4.21	0.60	3.07	(41.44, 57.72)

increasing θ. This is because when the tilting distribution predominates, the dispersion is increased which makes estimates wider and more conservative. For v, higher values correspond to assigning more weight to shorter, more optimistic outcomes. Accordingly the means in Table 1 are monotonically decreasing in v. It is also somewhat surprising to note that the standard deviations also are decreasing in v (except for the case when $v = 0$ and $\theta = 1/4$). Inspection of Figure 5 indicates that the judgments rendered tend to be optimistic or neutral at worst (coincidentally, this is consistent with our review in Section 2). Hence a value of $v = 1/4$ further concentrates the optimistic nature of the judgments into the shorter times, reducing the standard deviation. Larger values of v counteract this, especially when θ is low.

Graphical displays of the distributions of total project times appear in Figure 6. The most conservative results for the distribution of total project time can be seen when $v = 1/4$ and $\theta = 1/4$. This distribution is the least skewed of all those displayed, and appears approximately uniform across the middle third of its range. We observe that the distribution in the centre of the top row for the beta rectangular when $\theta = 1/2$ is equivalent to the distribution for the tilted beta with $\theta = 1/2, v = 1/2$ in the centre of the third row, and these coincide as we would expect.

Figure 7 provides another way of viewing the changes in project times as a function of distributional parameters. It plots the CDF for the simulated project times under selected values of θ and v. On the left side where $\theta = 1/4$, the tilting portion of the mixture is predominant. We see there that the optimistic assessment of $v = 3/4$ leads to

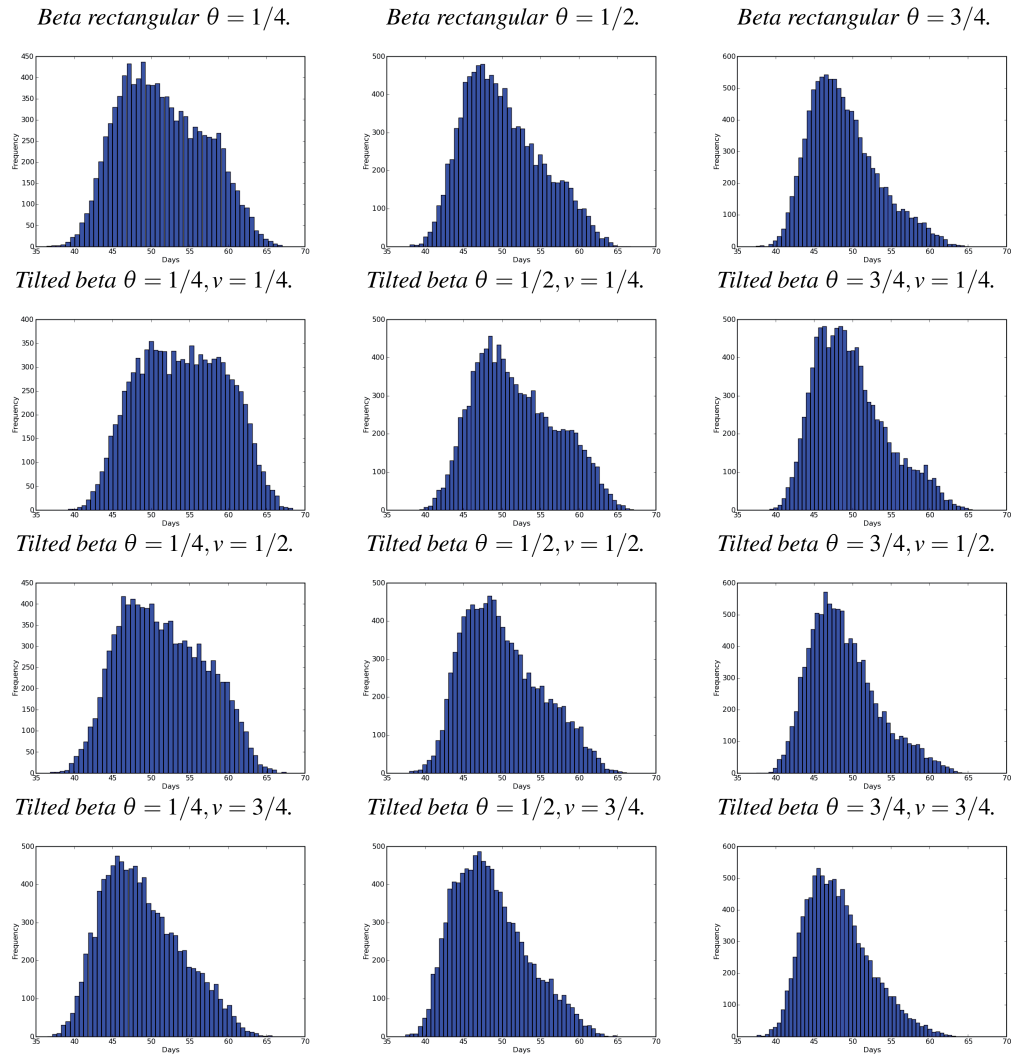


Figure 6: *Distributions of the total project time (Electronic Module Development Project).*

a relatively high cumulative probability of completion by 55 days. For less optimistic values of v, the cumulative probability of completion by 55 days (or other values we might select) falls off considerably. The right side of Figure 7 shows the CDF when the beta portion of the mixture predominates. The CDFs have some variability due to v but in general the CDFs are closer together and rise more steeply since they preserve more of the classic PERT beta influence. For completeness, we also observe that the CDF for the beta rectangular and the tilted beta with $v = 1/2$ again gives essentially the same result, as we would expect, since the solid beta rectangular line and the dashed tilted beta $v = 1/2$ line are essentially superimposed.

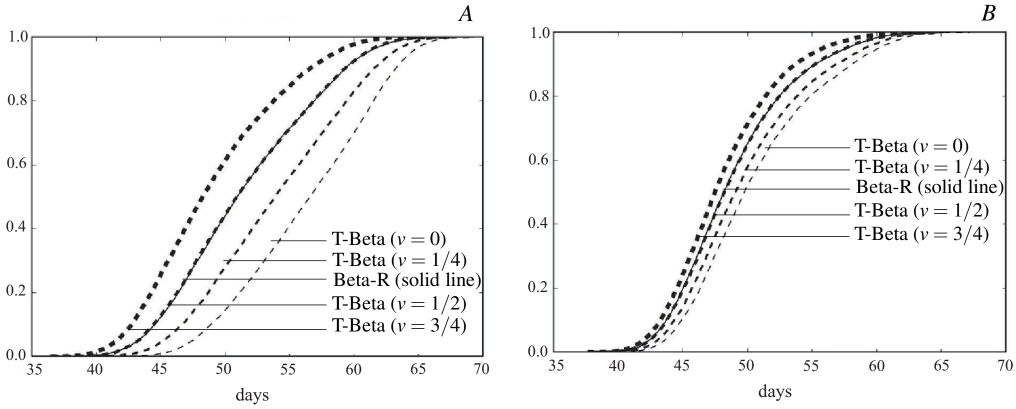


Figure 7: *Cumulative distribution function of beta rectangular distribution and tilted beta distribution for $\theta = 1/4$ (A) and $\theta = 3/4$ (B)*

Finally, Table 2 shows an example of what would constitute a rare or tail-area event under the different distributions. Here the 95% value of the CDF is obtained for v and θ taking on the values 0, 1/4, 1/2 and 3/4. With the introduction of the parameter v, the CDF of the tilted beta distribution may provide project time-to-completion estimates that can be either higher or lower than the beta rectangular distribution. The most striking comparison involves in the case when $\theta = 1$ which is the classic PERT beta case. We observe the classic PERT result would say there is a 95% chance of the project being completed by approximately 53.9 days, excluding some Monte Carlo error visible for the four values of v in the plot. Compare this results with the case of $\theta = 3/4$ where a small amount of extra-beta variability has been mixed in but the beta distribution still predominates at $\theta = 3/4$. Here even under the most optimistic case ($v = 3/4$) the 95% completion time has increased by 2 days to about 55.9 days. Hence, this example 'worst-case scenario' is two days worse than that given by the classic PERT beta. With less optimistic values of v, the time increases further. For example, with $v = 3/4$ and $\theta = 1/2$ we approach 57.7, i.e. approximately four days more than PERT.

Table 2: *The maximum time needed to complete the 95% the project for beta rectangular (Beta-R) and tilted beta (T-Beta) with $v = 0$, $v = 1/4$, $v = 1/2$ and $v = 3/4$.*

θ	Beta-R	T-Beta ($v = 0$)	T-Beta ($v = 1/4$)	T-Beta ($v = 1/2$)	T-Beta ($v = 3/4$)
0.25	60.98	**64.68**	63.06	60.95	**58.39**
0.50	59.58	**62.80**	61.45	59.60	**57.69**
0.75	57.75	**60.03**	59.22	**57.68**	55.98
1.00	53.90	**53.92**	**53.69**	53.78	53.83

In most cases, when the parameter v is higher than 1/2, the tilted beta distribution will be by construction more optimistic than the beta rectangular distribution and as a consequence its estimates will indicate that a lower time will be needed to complete the project. Conversely, when v is lower than 1/2, the tilted beta distribution provides more conservative estimations.

6. Conclusion

The introduction of different activity distributions has played an important part in the PERT methodology. However, this issue divides the researchers of this field. Some authors argue against the introduction of new probability distributions into PERT (see Clark, 1962; Hajdu, 2013; Hajdu and Bokor, 2014). Conversely as shown in Section 3 other authors have applied new distributions. Regarding the current paper, this debate has parallels in statistical practice. Some authors use robust statistical methods to handle outliers while other authors adopt less formal techniques or may even naively do nothing at all.

This paper introduced the tilted beta distribution and shown how it can be used in project management. Since the classic PERT results can be reproduced, it is simple to adopt in any environment where PERT is utilized. We can easily explain to executives and decision makers that incorporating additional uncertainty can help us to arrive at new insights. Elicitation of parameters is straightforward or one could perform sensitivity analysis using several parameter values as we have done here. The tilted beta has a number of attractive computational properties such as being easy to simulate from and having closed-form moment expressions. In summary, the tilted beta distribution provides project managers with a flexible and easy to work with distribution that allows for the extensive representation of optimistic or pessimistic beliefs regarding activity times.

Past work has pointed out the need to describe a more flexible distribution which allows for varying amounts of dispersion and greater likelihoods of more extreme tail-area events (Hahn, 2008). The construction of beta rectangular distribution is characterized by greater flexibility in the variance. However, this distribution assigns the same probability density at both the high and low extreme values. With the introduction of tilted beta distribution, we have expanded the set of continuous type distributions defined on a bounded domain, with the advantage of accommodating different relative likelihoods of high versus low extreme tail-area events and, as opposed to other distributions applied in this methodology, the tilted beta has an expression of the expected value where the extreme values have different weight. As a consequence, this distribution will be more relevant for modeling a broader range of heavy tailed phenomena. In a closely related work, the elevated two-sided power distribution (García et al., 2011) also permits different relative likelihoods of high versus low extreme tail-area events; however, as described above, the current distribution is simpler to use in practice. Furthermore, note that the procedure to induce tilting can be applied to any bounded continuous distribution.

We have compared the results of the tilted beta distribution with the results of beta rectangular distribution for different values of the parameter θ and ν. The results of the application show that this probabilistic model permits a risk manager to incorporate more optimistic and pessimistic scenarios than the beta rectangular distribution due to the flexibility of the tilted beta. Our literature review suggests that experts may tend to be too optimistic, and the beta distribution gives little weight to outliers in the standard PERT Type I and Type II beta conditions. The current methodology redresses these issues.

The distributions presented in this paper will be closer to the uniform since they give more weight to the tails. We think this gives some evidence that the distribution chosen is relevant by considering a recent paper by Hajdu and Bokor (2014). For larger projects, 10% can be the difference between a project being on time or late and the authors show that the uniform distribution is similar to PERT-beta + 10%. Furthermore, for small projects the authors state that it may not matter much. However, the incentive to use PERT with smaller projects is probably smaller.

There are at least two areas in which this research can be extended: first, the use of heavy-tailed distributions in the context of different activity calendars (Hajdu, 2013); second, to find more applications of these distributions fitting extreme tail-area events which are present in a great variety of fields such us finance, groundwater hydrology and atmospheric science among others.

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References

Bevilacqua, M., F. E. Ciarapica, and G. Giacchetta (2009). Critical chain and risk analysis applied to high-risk industry maintenance: a case study. *International Journal of Project Management*, 27, 419–432.

Boulding, W., R. Morgan, and R. Staelin (1997). Pulling the plug to stop new product drain. *Journal of Marketing Research*, 34, 164–176.

Chaloner, K. M. and G. T. Duncan (1983). Assessment of a beta prior distribution: PM Elicitation. *Journal of the Royal Statistical Society. Series D (The Statistician)*, 32, 174–180.

Clark, C. E. (1962). The PERT model for the distribution of an activity time. *Operations Research*, 10, 405–406.

Gallagher, C. (1987). A note on PERT assumptions. *Management Science*, 33, 1360.

García, C. B., J. García, and S. Cruz (2010). Proposal of a new distribution in PERT methodology. *Annals of Operations Research*, 181, 515–538.

García, C. B., J. García Pérez, and J. R. van Dorp (2011). Modeling heavy-tailed, skewed and peaked uncertainty phenomena with bounded support. *Statistical Methods and Applications*, 20, 463–482.

García Pérez, J., S. Cruz Rambaud, and C. B. García García (2005). The two-sided power distribution for the treatment of the uncertainty in PERT. *Statistical Methods & Applications*, 14, 209–222.

Garvey, P. R. (2000). *Probability Methods for Cost Uncertainty Analysis, a Systems Engineering Perspective*. New York, NY: Marcel Dekker.

Gillard, S. (2005). Managing IT projects: Communication pitfalls and bridges. *Journal of Information Science*, 31, 37–43.

Grubbs, F. E. (1962). Attempts to validate certain PERT statistics or 'Picking on PERT'. *Operations Research*, 10, 912–915.

Hahn, E. D. (2008). Mixture densities for project management activity times: a robust approach to PERT. *European Journal of Operational Management*, 188, 450–459.

Hajdu, M. (2013). Effects of the application of activity calendars on the distribution of project duration in PERT networks. *Automation in Construction*, 35, 397–404.

Hajdu, M. and O. Bokor (2014). The effects of different activity distributions on project duration in pert networks. *Procedia-Social and Behavioral Sciences*, 119, 766–775.

Herrerías, J. M., R. Herrerías, and J. R. van Dorp (2009). The generalized two-sided power distribution. *Journal of Applied Statistics*, 36, 573–587.

Herrerías, R. (1989). Utilización de modelos probabilísticos alternativos para el método PERT. Aplicación al análisis de inversiones. *Estudios de Economía Aplicada, Secretariado de Publicaciones de la Universidad de Valladolid*, 89–112.

Herrerías, R. and H. Calvete (1987). Una ley de probabilidad para el estudio de los flujos de caja de una inversión. *Libro Homenaje al Profesor Gonzalo Arnaiz Vellando*, 279–296. Madrid: ICE.

Hill, J., L. C. Thomas, and D. E. Allen (2000). Experts' estimates of task duration in software development projects. *International Journal of Project Management*, 18, 13–21.

Iacovou, C. L., R. L. Thompson, and H. J. Smith (2009). Selective status reporting in information systems projects: a dyadic-level investigation. *MIS Quarterly*, 33, 785–810.

Johnson, D. (1997). The triangular distribution as a proxy for the beta distribution in risk analysis. *The Statistician*, 46, 387–398.

Kamburowski, J. (1997). New validations of PERT times. *Omega*, 25, 323–328.

Keefer, D. L. and W. A. Verdini (1993). Better estimation of PERT activity time parameters. *Management Science*, 39, 1086–1091.

Keil, M., G. Depledge, and A. Rai (2007). Escalation: the role of problem recognition and cognitive bias. *Decision Sciences*, 38, 391–421.

Keil, M., J. Mann, and A. Rai (2000). Why software projects escalate: an empirical analysis and test of four theoretical models. *MIS Quarterly*, 24, 631–664.

Kotiah, T. C. T. and N. D. Wallace (1973). Another look at the PERT assumptions. *Management Science*, 20, 44–49.

Kotz, S. and J. R. van Dorp (2004). *Beyond Beta: Other Continuous Families of Distributions with Bounded Support*. Singapore: World Scientific.

Littlefield, Jr., T. K. and P. H. Randolph (1987). An answer to Sasieni's question on PERT times. *Management Science*, 33, 1357–1359.

López Martín, M. M., C. B. García, J. García, and M. A. Sánchez (2012). An alternative for robust estimation in project management. *European Journal of Operational Research*, 220, 443–451.

MacCrimmon, K. R. and C. A. Ryavec (1964). An analytical study of the PERT assumptions. *Operations Research*, 12, 16–37.

Malcolm, D. G., J. H. Roseboom, C. E. Clark, and W. Fazar (1959). Application of a technique for research and development program evaluation. *Operations Research*, 7, 646–669.

Megill, R. E. (1984). *An Introduction to Risk Analysis* (2nd ed.). Tulsa, OK: PennWell Books.

Moder, J. J., C. R. Phillips, and E. W. Davis (1983). *Project Management with CPM, PERT and Precedence Diagramming* (3rd ed.). New York: Van Nostrand Reinhold Company Inc.

Moder, J. J. and E. G. Rodgeres (1968). Judgement estimates of the moments of PERT type distributions. *Management Science*, 15, B76–B83.

Mohan, S., M. Gopalakrishnan, H. Balasubramanian, and A. Chandrashekar (2007). A lognormal approximation of activity duration in pert using two time estimates. *The Journal of the Operational Research Society*, 58, 827–831.

Nadarajah, S. and A. Gupta (2004). Products and linear combinations. In A. Gupta and S. Nadarajah (Eds.), *Handbook of Beta Distribution and Its Applications*, 55–88. New York: Marcel Dekker.

Öztaş, A. and O. Ökmen (2005). Judgmental risk analysis process development in construction projects. *Building and Environment*, 40, 1244–1254.

Pham-Gia, T. and N. Turkkan (1993). Bayesian analysis of the difference of two proportions. *Communications in Statistics-Theory and Methods*, 22, 1755–1771.

Pouliquen, L. Y. (1970). *Risk Analysis in Project Appraisal*. Baltimore: John Hopkins University Press.

Powell, M. R. and J. D. Wilson (1997). *Risk Assessment for National Natural Resource Conservation Programs*. Washington: Resources for the Future.

Romero, C. (1991). *Técnicas de Programación y Control de Proyectos*. Madrid: Pirámide.

Sasieni, M. W. (1986). A note on PERT times. *Management Science*, 32, 1652–1653.

Sengupta, K., T. K. Abdel-Hamid, and L. N. van Wassenhove (2008). The experience trap. *Harvard Business Review*, 86, 94–101.

Smith, H. J., M. Keil, and G. Depledge (2001). Keeping mum as the project goes under: toward an explanatory model. *Journal of Management Information Systems*, 18, 189–227.

Snow, A. P., M. Keil, and L. Wallace (2007). The effects of optimistic and pessimistic biasing on software project status reporting. *Information & Management*, 44, 130–141.

Suárez, A. (1980). *Decisiones Óptimas de Inversión y Financiación en la Empresa*. Madrid: Pirámide.

Topp, C. W. and F. C. Leone (1955). A family of J-Shaped frequency functions. *Journal of the American Statistical Association*, 50, 209–219.

Trietsch, D., L. Mazmanyan, L. Gevorgyan, and K. R. Baker (2012). Modeling activity times by the Parkinson distribution with a lognormal core: theory and validation. *European Journal of Operational Research*, 216, 386–396.

Vãduva, I. (1971). Computer generation of random variables and vectors related to PERT problems. *Proceedings of the 4th conference on Probability Theory*, Brasov, 381–395.

Van Dorp, J. R. and S. Kotz (2006). Modeling income distributions using elevated distributions. In R. Herrerías Pleguezuelo, J. Callejón Céspedes, and J. M. Herrerías Velasco (Eds.), *Distribution Models Theory*, 1–25. Singapore; 2006: World Scientific.

Williams, T. M. (1992). Practical use of distributions in network analysis. *Journal of the Operations Research Society*, 43, 265–270.